A Simple NK Model in MATLAB

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June 24, 2008

AGENDA

- **Gali's Textbook Model**
- **Calibration Code**
- **Solution Code**
- **Exercises**
  - Sensitivity Analysis
  - Recovering the Taylor Rule

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$$\pi_t = \beta \pi_{t+1|t} + \kappa x_t$$

$$i_t = \overline{rr}_t + \Delta x_{t+1|t} + \pi_{t+1|t}$$

$$i_t = \rho + \phi_x x_t + \phi_\pi \pi_t$$

NK MODEL WITH SIMPLE TAYLOR RULE
Gali (2003, Section 5.3)

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Potential output / natural rates are:

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Exogenous "supply" Δa_t and "demand" g_t

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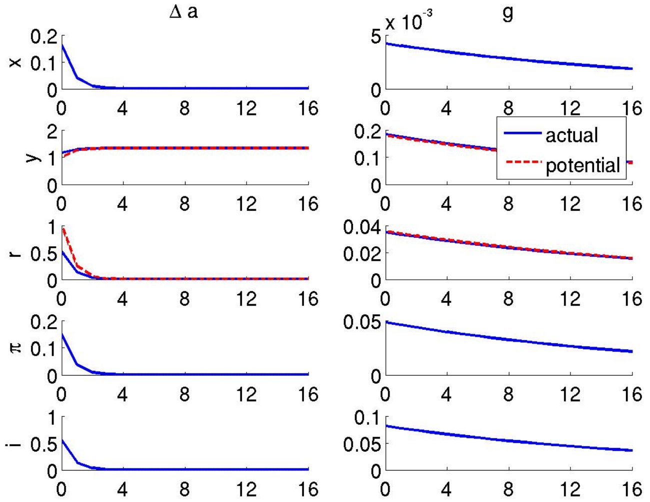
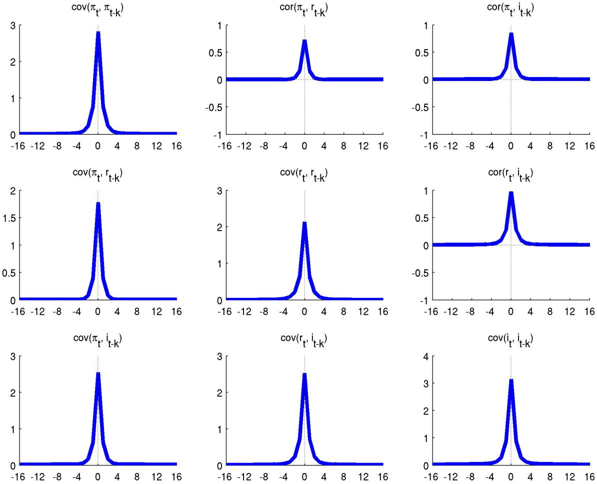
OUTPUT: IMPULSE RESPONSES



OUTPUT: AUTOCOVARIANCES



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- Gali gives us the following solution:

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \omega_a[\boldsymbol{I} - \rho_a \boldsymbol{A}_T]^{-1}\boldsymbol{B}_T \Delta a_t + \omega_g[\boldsymbol{I} - \rho_g \boldsymbol{A}_T]^{-1}\boldsymbol{B}_T g_t$$

(matrices defined in the paper and here ...)

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- To get i_t, plug the above into Taylor Rule

DERIVATION

After substituting TY into IS:

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \boldsymbol{A}_T \begin{bmatrix} x_{t+1|t} \\ \pi_{t+1|t} \end{bmatrix} + \boldsymbol{B}_T \overline{rr}_t$$

DERIVATION

BACKUP

Key Observations

- There are only forward looking variables
- Taylor principle ensures $\boldsymbol{A}_T$ is "stable"

Linear Difference System:

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \boldsymbol{A}_T \begin{bmatrix} x_{t+1|t} \\ \pi_{t+1|t} \end{bmatrix} + \boldsymbol{B}_T \overline{rr}_t$$

$$= \sum_{k=0}^{\infty} \boldsymbol{A}_T^k \boldsymbol{B}_T \, \overline{rr}_{t+k|t}$$

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$$= \sum_{k=0}^{\infty} \boldsymbol{A}_T^k \boldsymbol{B}_T \begin{bmatrix} \omega_a & \omega_g \end{bmatrix} \begin{bmatrix} \rho_a^k & 0 \\ 0 & \rho_g^k \end{bmatrix} \begin{bmatrix} \Delta a_t \\ g_t \end{bmatrix}$$

$$= \omega_a [\boldsymbol{I} - \rho_a \boldsymbol{A}_T]^{-1} \boldsymbol{B}_T \Delta a_t + \omega_g [\boldsymbol{I} - \rho_g \boldsymbol{A}_T]^{-1} \boldsymbol{B}_T g_t$$

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CALIBRATION VALUES

BACKUP

Private Sector Parameters		
β	0.99	Time preference
σ	1.00	Risk Aversion / Inverse EIS
θ	0.75	Calvo Probability of not repricing
κ	0.1717	Slope of Phillips Curve
ϕ	1.00	Inverse Frisch Labor Elasticity
Taylor Rule Parameters		
ϕ_π	1.50	Response to inflation
ϕ_x	0.50	Response to output gap
Driving Processes		
ρ_a	0.2500	Persistence of technology growth
σ_a	1.0000	Volatility of technology growth
ρ_g	0.9500	Persistence in g shocks
σ_g	0.3600	Volatility of g shocks

CALIBRATION
Gali p. 165f, p. 180

baselinecalibration.m

```
% preferences
beta     = .99;
phi      = 1;
sigma    = 1;
theta    = .75;
gamma    = 1;
% Taylor Rule
phi_pi   = 1.5;
phi_x    = .5;
% Driving Processes
rho_a    = 0.25;
sig_a    = .01 * 100;     % percentages
% etc ...
```

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EXERCISE

Write a program which . . .

1. defines a variable for each model parameter and sets it to its baseline value (or use `baselinecalibration.m`)
2. computes a matrix $\boldsymbol{C_{x\pi}}$ such that

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \boldsymbol{C_{x\pi}} \begin{bmatrix} \Delta a_t \\ g_t \end{bmatrix}$$

   (What are the dimensions of $\boldsymbol{C_{x\pi}}$, $\boldsymbol{A_T}$ and $\boldsymbol{B_T}$?)
3. computes a matrix $\boldsymbol{C_i}$ such that

$$i_t = \boldsymbol{C_i} \begin{bmatrix} \Delta a_t \\ g_t \end{bmatrix}$$

4. How does this solution depend on ρ_a and σ_a?

SYSTEM MATRICES
Gali p. 157, 159, 178, 179

(Return to Solution)

$$\boldsymbol{A}_T = \Omega \begin{bmatrix} \sigma & (1-\beta\phi_\pi) \\ \sigma\kappa & \kappa + \beta(\sigma + \phi_x) \end{bmatrix}$$

$$\boldsymbol{B}_T = \Omega \begin{bmatrix} 1 \\ \kappa \end{bmatrix}$$

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where . . .

$$\psi_a = \frac{1+\phi}{\sigma+\phi} \qquad \omega_a = \sigma\psi_a\rho_a$$

$$\psi_g = \frac{\sigma}{\sigma+\phi} \qquad \omega_g = \sigma(1-\psi_g)(1-\rho_g)$$

$$\kappa = (\sigma+\phi)\frac{(1-\theta)\cdot(1-\beta\theta)}{\theta} \qquad \Omega = 1/(\sigma+\phi_x+\kappa\phi_\pi)$$

COMPUTING THE RE SOLUTION

Linear Difference System:

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \boldsymbol{A}_T \begin{bmatrix} x_{t+1|t} \\ \pi_{t+1|t} \end{bmatrix} + \boldsymbol{B}_T \overline{rr}_t$$

$$= \sum_{k=0}^{\infty} A_T^k B_T \underbrace{\begin{bmatrix} \omega_a & \omega_g \end{bmatrix} \begin{bmatrix} \rho_a^k & 0 \\ 0 & \rho_g^k \end{bmatrix}}_{C_T^k} \begin{bmatrix} \Delta a_t \\ g_t \end{bmatrix}$$

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$$= \sum_{\boldsymbol{k=0}}^{\boldsymbol{\infty}} \boldsymbol{A}_{\boldsymbol{T}}^{\boldsymbol{k}} \boldsymbol{B}_{\boldsymbol{T}} \boldsymbol{C}_{\boldsymbol{T}}^{\boldsymbol{k}} \begin{bmatrix} \Delta a_t \\ g_t \end{bmatrix}$$

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$$= \sum_{k=0}^{\infty} \boldsymbol{A}_T^k \boldsymbol{B}_T \boldsymbol{C}_T^k \begin{bmatrix} \Delta a_t \\ g_t \end{bmatrix}$$

How to solve this in general?

- Brute force: iterate until convergence
- Smart force: iterate w/doubling algorithm
- closed form: Sylvester equation

Mathematics:

$$S = \sum_{k=0}^{\infty} A^k C (B')^k = ASB' + C$$

$$\Rightarrow \text{vec}\,(S) = [I - B \otimes A]^{-1}\,\text{vec}\,(C)$$

Note: When $A = B$ this is a.k.a. Lypunov equation

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MATLAB

```
vecS   = (I - kron(B, A)) \ C(:);
S      = reshape(vecS, rows, cols);
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Better: `dlyap` (control toolbox) or `disclyap` adapted from Hansen & Sargent's `doublej`

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Sensitivity Analysis

1. Call the interest rate response to Δa_t C_{ia}
2. Change ρ_a from 0 to 0.5 (with increments of 0.01) and plot C_{ia} as a function of ρ_a
   Hint: You could package the previous solution into a function
3. Change θ from 0.1 to 1 (with increments of 0.1) and plot C_{ia} as a function of θ
4. Plot C_{ia} as a function of ρ_a and θ
   Hints:
   - Store your results in a 2D matrix `Cia`
   - See `doc surf`
5. Do the same for the sensitivty to g shocks

Gali's Solution as function

```
function [Cx,Cpi,Ci] = funcNK(rho_a,theta)
% returns impact coefficients for simple Taylor
    Rule Model of Gali, Section 5.3
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% returns impact coefficients for simple Taylor
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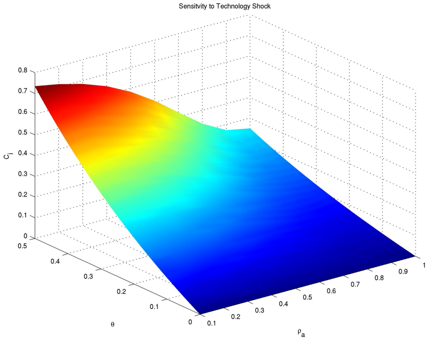
Loop over grid of values

```
Ci = repmat(NaN, [length(RHOS), ...
 length(THETAS), 2]);
for r = 1 : length(RHOS)
   for t = 1 : length(THETAS)
      [cx, cpi, Ci(r,t,:)] = ...
          funcNK(RHOS(r), THETAS(t));
   end
end
```

3D Plot

```
figure
surf(THETAS, RHOS, Ci(:,:,1))
shading interp
```



Using Gali's calibration

1. Simulate $T = 100$ observations for Δa_t and g_t (`AR1sim`)
2. Compute observations for x_t, π_t and i_t
3. Estimate

$$i_t = \phi + \phi_\pi \pi_t + \phi_x x_t + \varepsilon_t$$

EXERCISE

Estimate Taylor rule in simulated NK model

Using Gali's calibration

1. Simulate $T = 100$ observations for Δa_t and g_t (`AR1sim`)
2. Compute observations for x_t, π_t and i_t
3. Estimate
$$i_t = \phi + \phi_\pi \pi_t + \phi_x x_t + \varepsilon_t$$
4. (Discuss your results in light of Cochrane (2007, NBER))
5. What would happen if you were to repeat the regressions above ...
   - ... whilst setting $g_t = 0$ for all observations?
   - ... with $T = 10,000$ observations?