Solving Linear DSGE Models w/MATLAB

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AGENDA

- **RE Solution for Linear Systems**
- **State Space Representation**
- **Application: Interest Rate Smoothing**
- **More Extensions of NK Model**

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LINEAR DIFFERENCE SYSTEM

Setup

$$\boldsymbol{\mathcal{A}}\ \boldsymbol{x}_{t+1|t} = \boldsymbol{\mathcal{B}}\ \boldsymbol{x}_t + \boldsymbol{\mathcal{C}}\ \boldsymbol{z}_t$$

- $\boldsymbol{x}_t$ endogenous, $\boldsymbol{z}_t$ are exogenous
- Typically from linearized FOC

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- Typically from linearized FOC
- Backward vs forward looking variables ("k_t vs d_t")

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$$\boldsymbol{z}_{t+1} = \Phi \boldsymbol{z}_t + Q w_{t+1}$$

- $\boldsymbol{x}_t$ endogenous, $\boldsymbol{z}_t$ are exogenous
- Typically from linearized FOC
- Backward vs forward looking variables ("k_t vs d_t")
- VAR(1) for $\boldsymbol{z}_t$ (companion form, w/o loss of generality)

Backward looking (Klein):

k_0 given and exogenous forecast errors

$$k_{t+1} = k_{t+1|t} + Q_k w_{t+1}$$

e.g. technology process, capital, consumption habit, called “predetermined” by Svensson

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Predetermined (BK/KW)

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Predetermined (BK/KW)

k_0 given and zero forecast errors

$$k_{t+1} = k_{t+1|t}$$

Forward looking when not backward looking, a.k.a. "jump" variables, e.g. consumption, asset prices

For the NK baseline model ...

W/o substituting the Taylor rule into the IS curve:

1. What are $\boldsymbol{x}$ and $\boldsymbol{z}$?
2. How many predetermined variables are there?
3. Setup matrices $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{C}}$, Φ and Q

NK MODEL WITH SIMPLE TAYLOR RULE

Gali (2003, Section 5.3)

BACKUP

New Keynesian textbook model:

$$\begin{aligned} \pi_t &= \beta \pi_{t+1|t} + \kappa x_t \\ i_t &= \overline{rr}_t + \Delta x_{t+1|t} + \pi_{t+1|t} \\ i_t &= \phi_x x_t + \phi_\pi \pi_t \end{aligned}$$

Potential output / natural rates are:

$$\begin{aligned} \bar{y}_t &= \psi_a a_t + \psi_g g_t \\ \overline{rr}_t &= \omega_a \Delta a_t + \omega_g g_t \end{aligned}$$

Exogenous "supply" Δa_t and "demand" g_t

$$\begin{aligned} \Delta a_{t+1} &= \rho_a \Delta a_t + \sigma_a \varepsilon^a_{t+1} \\ g_{t+1} &= \rho_g g_t + \sigma_g \varepsilon^g_{t+1} \end{aligned}$$

BASIC NK MODEL: RE SYSTEM

FOC System: $(N_k = 0)$

$$\underbrace{\begin{bmatrix} 0 & 0 & \beta \\ 0 & \sigma & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{\mathcal{A}}} \underbrace{\begin{bmatrix} i_{t+1|t} \\ x_{t+1|t} \\ \pi_{t+1|t} \end{bmatrix}}_{\boldsymbol{x}_{t+1|t}} = \underbrace{\begin{bmatrix} 0 & -\kappa & 1 \\ 1 & \sigma & 0 \\ -1 & \phi_x & \phi_\pi \end{bmatrix}}_{\boldsymbol{\mathcal{B}}} \begin{bmatrix} i_t \\ x_t \\ \pi_t \end{bmatrix} - \underbrace{\begin{bmatrix} 0 & 0 \\ \omega_a & \omega_g \\ 0 & 0 \end{bmatrix}}_{\boldsymbol{\mathcal{C}}} \boldsymbol{z}_t$$

Exogenous Drivers:

$$\underbrace{\begin{bmatrix} \Delta a_{t+1} \\ g_{t+1} \end{bmatrix}}_{\boldsymbol{z}_{t+1}} = \underbrace{\begin{bmatrix} \rho_a & 0 \\ 0 & \rho_g \end{bmatrix}}_{\Phi} \boldsymbol{z}_t + \underbrace{\begin{bmatrix} \sigma_a & 0 \\ 0 & \sigma_g \end{bmatrix}}_{Q} w_{t+1}$$

LINEAR RE SOLUTION

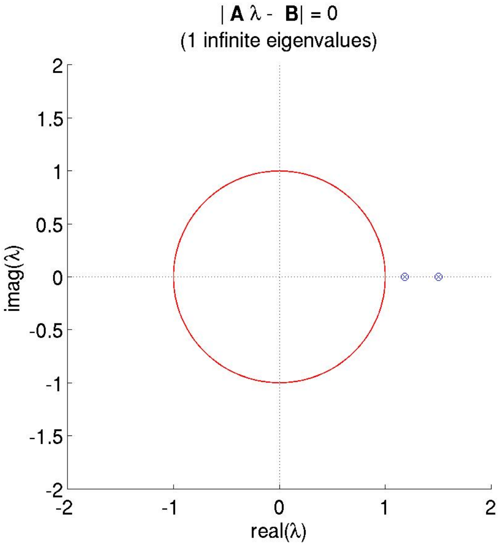
$$\mathcal{A}\, x_{t+1|t} = \mathcal{B}\, x_t + \mathcal{C}\, z_t$$

Determinacy condition: $|\mathcal{A}\,\lambda - \mathcal{B}\,| = 0$

- must have N_k stable roots ($|\lambda| < 1$)
- static equations add infinite roots ("unstable")
- `eig(BB, AA)` and `lambda=ploteigenvalues(AA, BB)`

EIGENVALUE PLOT

`ploteigenvalues` for simple NK model



For the simple Taylor rule model . . .

1. Check the determinacy condition.
   (Recall that all roots of $|\boldsymbol{\mathcal{A}}z - \boldsymbol{\mathcal{B}}| = 0$ must lie outside the unit circle in this model.)

2. Vary ϕ_π and verify whether the determinacy condition gets violated by checking `eig(BB,AA)`

LINEAR RE SOLUTION

$$\boldsymbol{\mathcal{A}}\ \boldsymbol{x}_{t+1|t} = \boldsymbol{\mathcal{B}}\ \boldsymbol{x}_t + \boldsymbol{\mathcal{C}}\ \boldsymbol{z}_t$$

$$k_{t+1|t} = \boldsymbol{P}k_t + \boldsymbol{L}\boldsymbol{z}_t$$

$$d_t = \boldsymbol{F}k_t + \boldsymbol{N}\boldsymbol{z}_t$$

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LINEAR RE SOLUTION

$$\begin{aligned} \boldsymbol{\mathcal{A}}\, \boldsymbol{x}_{t+1|t} &= \boldsymbol{\mathcal{B}}\, \boldsymbol{x}_t + \boldsymbol{\mathcal{C}}\, \boldsymbol{z}_t \\ k_{t+1} &= \boldsymbol{P} k_t + \boldsymbol{L} \boldsymbol{z}_t + Q_k w_{t+1} \\ d_t &= \boldsymbol{F} k_t + \boldsymbol{N} \boldsymbol{z}_t \end{aligned}$$

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Solution Code:

```
[f,p,n,l] = solabc(AA, BB, Nk , CC, Phi)
```

(adapted from Paul Klein's `[f,p] = solab(AA, BB, Nk)`)

SOLAB VS SOLABC
Klein (2000, JEDC)

- `solabc.m` has been written by me based on Klein's paper
- (Klein's own toolbox contains only `solab.m`)
- You can always replace
  `solabc(AA, BB, Nk , CC, Phi)` by
  `solab(AAnew, BBnew, Nk+Nz)` where

$$\mathbf{A}^{\text{new}} = \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{\mathcal{A}} \end{bmatrix}$$

$$\mathbf{B}^{\text{new}} = \begin{bmatrix} \Phi & 0 \\ \boldsymbol{\mathcal{C}} & \boldsymbol{\mathcal{B}} \end{bmatrix}$$

  and $k_t^{\text{new}} = \begin{bmatrix} \boldsymbol{z}_t & k_t \end{bmatrix}'$
- But solving large system is inefficient
- ...particularly for MLE, Monte Carlo etc

Using solabc . . .

1. Solve the NK baseline model
   (w/o substituting Taylor rule into the IS curve)
2. Compare the results with your previous calculation
3. Do the same using `solab.m`? What is different?

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Recall: Generic State Space

State dynamics: $X_{t+1} = \boldsymbol{A} X_t + \boldsymbol{B} w_{t+1} \quad (w_{t+1} \sim iid(0, \boldsymbol{I}))$

Observables: $Y_t = \boldsymbol{C} X_t$

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RE Solution

$$\underbrace{\begin{bmatrix} \boldsymbol{z}_{t+1} \\ k_{t+1} \end{bmatrix}}_{X_{t+1}} = \underbrace{\begin{bmatrix} \Phi & 0 \\ \boldsymbol{L} & \boldsymbol{P} \end{bmatrix}}_{\boldsymbol{A}} X_t + \underbrace{\begin{bmatrix} Q \\ Q_k \end{bmatrix}}_{\boldsymbol{B}} w_{t+1}$$

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$$Y_t = \begin{bmatrix} d_t \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{N} & \boldsymbol{F} \\ \vdots & \ddots \end{bmatrix}}_{\boldsymbol{C}} X_t$$

Using solabc ...

1. Solve the NK baseline model
   (w/o substituting Taylor rule into the IS curve)
2. Compare the results with your previous calculation
3. Do the same using `solab.m`? What is different?
4. **Compute state space matrices A, B, C**

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EXERCISE

Implement interest rate smoothing ...

$$i_t = \phi_i i_{t-1} + \phi_x x_t + \phi_\pi \pi_t$$

1. How would you setup the RE system, and what is $\boldsymbol{x_t}$?
2. How many predetermined variables are there?

INTEREST RATE SMOOTHING

New

$$i_t = \phi_i i_{t-1} + \phi_x x_t + \phi_\pi \pi_t$$
$$N_k = 1$$

RE System:

$$\mathcal{A} \begin{bmatrix} i_t \\ x_{t+1|t} \\ \pi_{t+1|t} \end{bmatrix} = \mathcal{B} \begin{bmatrix} i_{t-1} \\ x_t \\ \pi_t \end{bmatrix} + \mathcal{C} \begin{bmatrix} \Delta a_t \\ g_t \end{bmatrix}$$

INTEREST RATE SMOOTHING

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$$i_t = \phi_i i_{t-1} + \phi_x x_t + \phi_\pi \pi_t$$
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RE System:

$$\begin{bmatrix} 0 & 0 & \beta \\ -1 & \sigma & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_t \\ x_{t+1|t} \\ \pi_{t+1|t} \end{bmatrix} = \begin{bmatrix} 0 & -\kappa & 1 \\ 0 & \sigma & 0 \\ \phi_i & \phi_x & \phi_\pi \end{bmatrix} \begin{bmatrix} i_{t-1} \\ x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\omega_a & -\omega_g \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta a_t \\ g_t \end{bmatrix}$$

Implement interest rate smoothing ...

$$i_t = \phi_i i_{t-1} + \phi_x x_t + \phi_\pi \pi_t$$

1. How would you setup the RE system, and what is $\boldsymbol{x_t}$?
2. How many predetermined variables are there?
3. Solve the extended model and compute its state space
4. Set $\phi_i = 0$ and compare the results with your previous computations
5. Plot impulse responses

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Inflation Indexation

$$\pi_t - \gamma\pi_{t-1} = \beta(\pi_{t+1|t} - \gamma\pi_t) + \kappa x_t$$

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Cost Push Shock

$$\pi_t - \gamma\pi_{t-1} = \beta(\pi_{t+1|t} - \gamma\pi_t) + \kappa x_t + u_t$$

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Cost Push Shock (add u_t to $\boldsymbol{z}_t$)

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Cost Push Shock (add u_t to $\boldsymbol{z}_t$)

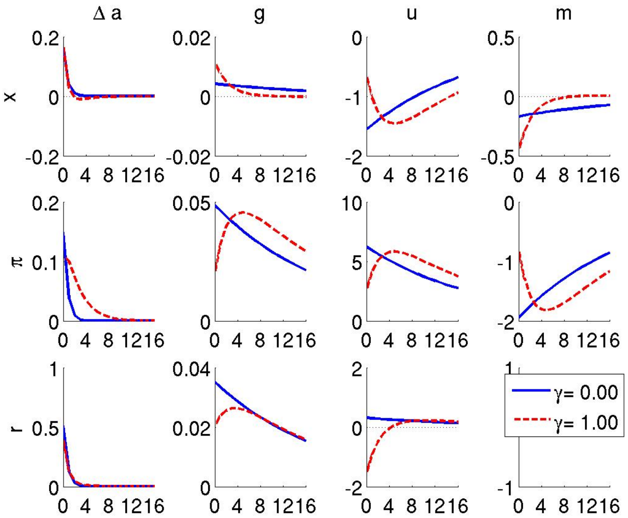
$$\pi_t - \gamma\pi_{t-1} = \beta(\pi_{t+1|t} - \gamma\pi_t) + \kappa x_t + u_t$$

Policy Shock (add m_t to $\boldsymbol{z}_t$)

$$i_t = \phi_i i_{t-1} + \phi_x x_t + \phi_\pi \pi_t + m_t$$

FULL VS ZERO INDEXATION

$\gamma = 0$ and $\gamma = 1$ in baseline model w/o IR smoothing



VARIATIONS IN INDEXATION RATE

IRF of i_t to u_t (baseline model w/o IR smoothing)