Memo: **VECM Notes**
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1 Converting VECM into State Space and MA representation

Let S_t be a vector of non-stationary, $I(1)$ variables, which are cointegrated such that $\alpha' S_t \sim I(0)$ for some matrix of cointegrating vectors α. There is then a VECM representation

$$\Delta S_t = F \Delta S_{t-1} + G(\alpha' S_{t-1}) + e_t \qquad e_t \sim iid(0, \Omega) \tag{1}$$

(This can be easily generalized to a higher lag order.)

The associated state-space representation is[1]

$$X_t = \begin{bmatrix} \Delta S_t \\ \alpha' S_t \end{bmatrix} = \begin{bmatrix} F & G \\ \alpha' F & I + \alpha' G \end{bmatrix} X_{t-1} + \begin{bmatrix} I \\ \alpha' \end{bmatrix} e_t \tag{2}$$

$$= \mathcal{A} X_{t-1} + \mathcal{B} e_t \tag{3}$$

from which follows the VMA representation

$$\Delta S_t = \begin{bmatrix} I & 0 \end{bmatrix} \left(I - \mathcal{A} L\right)^{-1} \mathcal{B} e_t = C(L) e_t \tag{4}$$

where L is the lag operator. Due to the cointegrating relationship, the matrix of long-run coefficients $C(1)$ is singular, in particular we have $\alpha' C(1) = 0$, since $\alpha' C(1)$ measures the long-run effect of a shock on the cointegrating vectors, which are stationary and thus their long-run responses are zero. (Section A below provides further details.)

[1]Notice that $\alpha' S_t = \alpha(\Delta S_t + S_{t-1})$.

2 Long-Run Shocks and the Innovation Variance

Suppose there is a one-to-one mapping between the VECM residuals e_t and structural shocks ε_t, $e_t = B\varepsilon_t$ which must obviously satisfy $BB' = \Omega$. Further suppose that some of these structural shocks shall have no effect on the levels S_t in the long-run. I will now derive the restrictions imposed by this long-run restriction on the impact coefficients B.

If some structural shocks shall have no impact on S_t in the long-run, the long-run coefficients, denoted $\Gamma(1) = C(1)B$, must obey the following zero restriction:

$$\Gamma(1) = \begin{bmatrix} X & 0 \end{bmatrix} \tag{5}$$

The restriction follows from the assumed cointegration, $\alpha' S_t \sim I(0)$.

A singular-value decomposition of $C(1)$ yields

$$C(1) = V \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} W' = V_1 S_1 W_1' \tag{6}$$

where $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ and $W = \begin{bmatrix} W_1 & W_2 \end{bmatrix}$ are conformably partitioned, unitary matrices, $VV' = I$ and $WW' = I$.

Without loss of generality, B can be written as the product of W and another matrix $\tilde{B}$. As will be seen next, the long-run restriction requires that $\tilde{B}$ is (block-) triangular:

$$\tilde{B} = \begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \end{bmatrix} = \begin{bmatrix} \tilde{B}_{11} & 0 \\ \tilde{B}_{21} & \tilde{B}_{22} \end{bmatrix} \tag{7}$$

The restriction $\tilde{B}_{12} = 0$ follows from (5) and (6), since it ensures that

$$W_1' B = \begin{bmatrix} x & 0 \end{bmatrix} \tag{8}$$

where x denotes an arbitrary, conformably-sized matrix.

$\tilde{B}$ factorizes $\tilde{\Omega} = W'\Omega W$, and one possible factorization satisfying the long-run restriction (5) is thus the Choleski factorization of $\tilde{\Omega}$. Furthermore, Potential permutations must retain the spans of the column blocks given the by the Choleski factorization.

For uniqueness, suppose further that the upper block of X in (5) is upper-triangular. It is then straightforward to show, that the first column block of B — the columns associated with the long-run shocks — is uniquely given by the first block of

$$B = W\,\mathrm{chol}\,(\tilde{\Omega}) \tag{9}$$

A $C(1)$ is singular

The singularity of $C(1)$ holds also in sample, for any point estimates of F, G and α — provided that $\mathcal{A}$ is stable, of course. Apart from appealing to $\alpha' S_t \sim I(0)$, this can also be verified by computing the partitioned inverse of $I - \mathcal{A}$:

$$(I - \mathcal{A})^{-1} = \begin{bmatrix} I_F & -G \\ -\alpha' F & \alpha' G \end{bmatrix}^{-1} = \begin{bmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{bmatrix} \tag{10}$$

The standard formulas for the inverse of a partitioned matrix imply in this case $M^{12} = -M^{11} G(\alpha' G)^{-1}$. Further, it follows that

$$C(1) = M^{11}\left(I - G(\alpha' G)^{-1}\alpha'\right) \tag{11}$$

And Sylvester's determinant theorem yields:

$$|C(1)| = |M^{11}||(\alpha' G)^{-1}||\,(\alpha' G - \alpha' G)\,| = 0 \tag{12}$$

Furthermore, it is straightforward to show that $\alpha' C(1) = 0$ for any point estimates of α, F and G. To see this, notice that

$$
\begin{aligned}
M^{11} &= \left((I-F) + G(\alpha' G)^{-1}\alpha' F\right)^{-1} \\
&= (I-F)^{-1} - (I-F)^{-1}G\left(\alpha' G + \alpha' F(I-F)^{-1}G\right)^{-1}\alpha' F(I-F)^{-1} \\
&= (I-F)^{-1} - (I-F)^{-1}G\left(\alpha'(I-F)^{-1}G\right)^{-1}\alpha' F(I-F)^{-1} \\
\Rightarrow \alpha' M^{11} &= \alpha'(I-F)^{-1} - \alpha' F(I-F)^{-1} \\
&= \alpha' \\
\Rightarrow \alpha' C(1) &= \alpha' M^{11}\left(I - G(\alpha' G)^{-1}\alpha'\right) \\
&= 0
\end{aligned}
$$