Memo: **Linear RE models: Klein vs. Sims for Linear RE Systems**

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This note briefly summarizes the solutions of Klein (2000) and Sims (2002) for linear rational expectations models.

Both schemes derive equilibrium laws of motion for a vector of variables, S_t, given an expectational linear difference system. In both cases the solution has the form

$$S_{t+1} = PS_t + D\varepsilon_{t+1}$$

where ε_{t+1} is a vector of exogenous shocks and P and D are computed by both schemes. Both cases agree on the shock loadings D but will generally disagree in some part about P. The difference in transition matrices P is however innocuous, and does not affect impulse response, and the computation of population moments $E(S_t S'_{t+j})$. Model simulation are only affected, when the simulations are not properly initialized (as described by Sims (2002)).

The remainder of this note is structured as follows. Section 1 describes the describes the class of linear models considered here, which is the same as in Klein (2000). Section 2 reviews Klein (2000)'s solution and Section 3 derives the corresponding solution using the approach described by Sims (2002). Section 4 briefly compares both methods.

1 Class of Linear RE Models

As in Klein (2000), this note studies systems of the form

$$A\ E_t \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = B \begin{bmatrix} x_t \\ y_t \end{bmatrix} \qquad\qquad x_{t+1} - E_t x_{t+1} = C_\varepsilon \varepsilon_{t+1} \tag{1}$$

where x_t are "backward-looking" variables and y_t are "forward-looking" (a.k.a. "jump") variables. Shocks to backward-looking variables, ε_t, are purely exogenous.[1] The matrix C_ε allows for cases where the number of shocks, ε_t differs from the number of backward-looking variables (for example, when some of the backward-looking variables are tracking lagged values of x_t or y_t. Without loss of generality, shocks can be normalized such that $E(\varepsilon_t \varepsilon'_t) = I$ and $\mathrm{Var}_t\,(x_{t+1}) = C_\varepsilon C'_\varepsilon$.[2].

Similar systems have also been studied by Blanchard and Kahn (1980) and King and Watson (1998). While, King and Watson (1998) and Klein (2000), allow for A to be singular, the approach

[1] For the sake of exposition — and without loss of generality — no distinction is made here between exogenous forcing variables and endogenous state variables. Since x_t can always be partitioned into $\begin{bmatrix} s'_t & z'_t \end{bmatrix}'$ where $z_t = Fz_{t-1} + \varepsilon^z_t$ is a purely exogenous process and s_t tracks any remaining, endogenous state variables of the system. For computational efficiency, it is however advisable to solve directly for the dynamics of s_t (instead of x_t) and use the given process for z_t, see, for example, the discussion by Klein (2000).

[2] This is only relevant for the computation of second moments.

of Blanchard and Kahn (1980) requires A to be non-singular. This matters, since a singular A can, for example, arise, when some of the system's equations are purely static, involving only time t variables. King and Watson (1998) and Klein (2000) pursue different implementations of the same solution. While King and Watson (1997) pursue a system reduction to solve such singular systems, Klein (2000) uses a generalized complex Schur decomposition ("QZ decomposition").[3]

The approach of Sims (2002) is somewhat more general, but also nests systems as in (1). The key distinction is that the setup used in Sims (2002), does not require to partition variables into "backward-looking" and "forward-looking" variables. For the sake of comparison, this assumption will however be maintained here.

2 Klein (2000)

As described by Klein (2000) — and consistent with the conditions stated in Blanchard and Kahn (1980), King and Watson (1997) and Sims (2002) – the system has a unique, bounded solution, when the number of forward-looking variables is identical to the number of generalized eigenvalues of A and B that lie outside the unit circle.[4] When this condition is satisfied, there is a unique solution.[5]

Overview

The solution provides a direct mapping from backward-looking into forward-looking variables, and the transition dynamics can be described entirely in terms of the backward-looking variables:

$$y_t = Gx_t \qquad\qquad x_{t+1} = Px_t + \varepsilon_{t+1} \tag{2}$$

The matrices G and P will be derived shortly below.

To compare the solutions of Klein (2000) and Sims (2002) below, it will be useful to derive the joint evolution of x_t and y_t implied by the RE solution:

$$S_{t+1} \equiv \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} P & 0 \\ G\,P & 0 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} C_\varepsilon \\ GC_\varepsilon \end{bmatrix} \varepsilon_{t+1} = \bar{P}S_t + \bar{D}\varepsilon_{t+1} \tag{3}$$

Second moments of x_t and y_t can be computed in two ways: First, the second moments of x_t can be computed from $\mathrm{Var}\,(x_t) = P\,\mathrm{Var}\,(x_t)P' + \mathrm{Var}\,(\varepsilon_t)$ and second (co-)moments of y_t follow from the linear dependence between y_t and x_t given by (2). Alternatively — and at slightly higher computational cost — one could also operate on the joint second moment matrix based on (3)

$$\mathrm{Var}\,(S_t) = \bar{P}\,\mathrm{Var}\,(S_t)\bar{P}' + \bar{D}\bar{D}' \tag{4}$$

Since both approaches enforce (2) at all times both yield the same results.

[3]King and Watson (1998) combine matrix pencil transformations with the Jordan decomposition — used also by Blanchard and Kahn (1980) for non-singular systems.

[4]The generalized eigenvalues of A and B are the roots λ of $|A\lambda - B|$.

[5]As discussed below, a further rank condition, which is typically innocuous, needs to hold as well.

Derivation

Klein (2000) and Sims (2002) use the complex generalized Schur decomposition of A and B.

$$QAZ = S \qquad\qquad QBZ = T$$

where Q and Z are complex unitary matrices and S and T are upper triangular matrices.[6]

The generalized eigenvalues of A and B are given by the ratios of the diagonal elements of S and T, $\lambda_i = {}^{t_{ii}}/_{s_{ii}}$ (for $s_{ii} \neq 0$.) Without loss of generality, the blocks of Q, Z, S, T are then ordered such that their upper blocks are associated with the stable eigenvalues of the system ($|\lambda_i| < 1$). The number of unstable eigenvalues is given by the number of elements where $|\lambda_i| \geq 1$ or where $s_{ii} = 0$.

To solve the system, (1) is conveniently rewritten in terms of "canonical" variables

$$\begin{bmatrix} \chi_t \\ \gamma_t \end{bmatrix} \equiv Z' \begin{bmatrix} x_t \\ y_t \end{bmatrix} \tag{5}$$

The canonical variables are a linear transformation of the original variables, which allows to characterize the transition dynamics via a triangular system, such that the dynamics of γ_t are uncoupled from χ_t.

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} E_t \begin{bmatrix} \chi_{t+1} \\ \gamma_{t+1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} \chi_t \\ \gamma_t \end{bmatrix} \tag{6}$$

Importantly, the dynamics of γ_t are characterized solely by unstable eigenvalues:

$$\gamma_t = T_{22}^{-1} S_{22} E_t \gamma_{t+1} = 0 \tag{7}$$

Imposing the usual transversality condition, (7) requires $\gamma_t = 0$, from which follows that the original variables are entirely described in terms of χ_t:

$$x_t = Z_{11}\chi_t \qquad\qquad y_t = Z_{12}\chi_t = Z_{12} Z_{11}^{-1} x_t \tag{8}$$

And it follows the transition for x_t is entirely described by the evolution of the (canonical) backward-looking variables:

$$\chi_{t+1} = S_{11}^{-1} T_{11} \chi_t + Z_{11}^{-1} \varepsilon_{t+1} \tag{9}$$

$$x_{t+1} = Z_{11} \left(S_{11}^{-1} T_{11} \right) Z_{11}^{-1} x_t + \varepsilon_{t+1} \tag{10}$$

In terms of the notation introduced above, we have have $G = Z_{12} Z_{11}^{-1}$ and $P = Z_{11} \left(S_{11}^{-1} T_{11} \right) Z_{11}^{-1}$. Since the eigenvalues of P correspond to the set of generalized eigenvalues of A and B that are stable, the transition of x_t (and thus y_t) is stable.

The non-singularity of Z_{11} is an additional rank condition required for the existence of the equilibrium.[7]

[6] Notice that when $A = I$ this reduces to $Q = Z'$ and $S = I$.

[7] Similar rank conditions are also needed by Blanchard and Kahn (1980), King and Watson (1998) and Sims (2002).

3 Sims (2002)

The approach of Sims (2002) is more general than the above in at least two aspects. First, his framework does not require to specify the distinction between backward-looking variables and forward-looking variables. Second, his method is explicitly focused on the determination of endogenous forecast errors. In order to compare his method with Klein (2000), I will, of course, retain the distinction between backward-looking and forward-looking variables.

Overview

Sims (2002) considers the role of endogenous forecast errors for the determination of $S_t = \begin{bmatrix} x_t' & y_t' \end{bmatrix}'$. Endogenous forecast errors are denoted by η_t where $y_t - E_{t-1}y_t = \eta_t$ and (1) can be written as

$$AS_{t+1} = BS_t + D_\varepsilon \varepsilon_{t+1} + D_\eta \eta_{t+1} \tag{11}$$

$$\text{where} \qquad D_\varepsilon = \begin{bmatrix} A_{xx} \\ A_{yx} \end{bmatrix} C_\varepsilon \qquad D_\eta = \begin{bmatrix} A_{xy} \\ A_{yy} \end{bmatrix}$$

This corresponds to the setting of Sims (2002), absent constants and with iid disturbances, where A_{xx}, A_{xy} etc. are partition the matrix A.[8]

Sims (2002) also employs the QZ decomposition and orders its matrix blocks as discussed in the case of Klein (2000) above. When the conditions for existence and determinacy are satisfied the solution described in Sims (2002) yields:

$$S_{t+1} = \tilde{P} S_t + \bar{D} \varepsilon_{t+1} \tag{12}$$

As stated above, I will henceforth also assume that S_t can be partitioned as in (1).

As will be shown below, $\bar{D}$in (12) is identical to $\bar{G}$ in (3), which was derived fromKlein (2000). While $\tilde{P}$ differs from $\bar{P}$ in (3), both (12) and (3) generate the same impulse responses and second moments, since the difference between $\tilde{P}$ and $\bar{P}$ lies in the nullspace of $\bar{D}$.[9]

Derivation

Similar to Klein (2000), the derivation of Sims (2002) starts by restating (12) in terms of canonical variables χ_t and γ_t as in (5) using the QZ decomposition of A and B.

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} \chi_{t+1} \\ \gamma_{t+1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} \chi_t \\ \gamma_t \end{bmatrix} + QA \begin{bmatrix} C_\varepsilon \varepsilon_{t+1} \\ \eta_{t+1} \end{bmatrix} \tag{13}$$

[8] Specifically, (11) corresponds to equation (1) in Sims (2002) setting has parameters to $\Gamma_0 = A$, $\Gamma_1 = B$, $C = 0$, $\Psi = D_\varepsilon$ and $\Pi = D_eta$ where his $y(t)$, $z(t)$ and $\eta(t)$ corresponds to S_t, ε_t and η_t here. Note further that the specific relationship between D_ε, D_η and blocks of A assumed above, is a direct consequence of the explicit partitioning of S_t into x_t and y_t. As discussed above, serially correlated ε_{t+1} can always be mapped into this system by suitably augmenting S_t.

[9] As noted by (Sims, 2002, page 8), model simulations need to impose $y_0 = Gx_0$ (in my notation), which also accounts for the difference in the transition matrices.

As before, the dynamics of γ_t are characterized solely by unstable eigenvalues, which now imposes a specific relationship between the exogenous shocks, ε_t, and the endogenous forecast errors, η_t.

$$\gamma_t = T_{22}^{-1} S_{22} E_t \gamma_{t+1} - T_{22}^{-1} \left(Q_{21} A_{xx} + Q_{22} A_{yx} \right) C_\varepsilon \varepsilon_{t+1} - T_{22}^{-1} \left(Q_{21} A_{xy} + Q_{22} A_{yy} \right) \eta_{t+1} \quad (14)$$

Since $\gamma_t = E_t \gamma_t = 0$ it follows that

$$\left(Q_{21} A_{xx} + Q_{22} A_{yx} \right) C_\varepsilon \varepsilon_{t+1} + \left(Q_{21} A_{xy} + Q_{22} A_{yy} \right) \eta_{t+1} = 0$$

and thus $\eta_{t+1} = G C_\varepsilon \varepsilon_t$ where $G = Z_{12} Z_{11}^{-1}$.[10]

In order to remove the dependence on the endogenous forecast errors from the top row of (13), Sims (2002) pre-multiplies (13) by $\begin{bmatrix} I & -\Phi \end{bmatrix}$, where Φ is constructed such that $Q_{11} A_{xy} + Q_{12} A_{yy} = \Phi \left(Q_{21} A_{xy} + Q_{22} A_{yy} \right)$.[11]

The evolution of the canonical variables is then described by the following system:

$$\begin{bmatrix} \chi_{t+1} \\ \gamma_{t+1} \end{bmatrix} = \begin{bmatrix} S_{11}^{-1} T_{11} & S_{11}^{-1} \left(T_{12} - \Phi T_{22} \right) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \chi_t \\ \gamma_t \end{bmatrix} + \begin{bmatrix} Z_{11}^{-1} \\ 0 \end{bmatrix} C_\varepsilon \varepsilon_{t+1} \quad (15)$$

$$\Rightarrow S_{t+1} = \underbrace{Z \begin{bmatrix} S_{11}^{-1} T_{11} & S_{11}^{-1} \left(T_{12} - \Phi T_{22} \right) \\ 0 & 0 \end{bmatrix} Z'}_{\tilde{P}} S_t + \underbrace{\begin{bmatrix} I \\ Z_{12} Z_{11}^{-1} \end{bmatrix}}_{\bar{D}} C_\varepsilon \varepsilon_{t+1} \quad (16)$$

4 Comparison and Variance Computation

The transition matrix $\tilde{P}$ derived in (16) from Sims (2002) differs from

$$\bar{P} = Z \begin{bmatrix} S_{11}^{-1} T_{11} & 0 \\ 0 & 0 \end{bmatrix} Z'$$

derived from Klein (2000) due to the term $S_{11}^{-1} \left(T_{12} - \Phi T_{22} \right)$ in the transition equation for χ_{t+1}. However, this difference is not consequential for impulse responses

$$\tilde{P}^j \bar{D} = \bar{P}^j \bar{D}$$

and second moments

$$\mathrm{Var}\left(S_t \right) = \sum_{j=0}^{\infty} \bar{P}^j \bar{D} \bar{D}' (\bar{P}^j)' \ .$$

To see this note that

$$\left(\tilde{P} - \bar{P} \right) \bar{D} = Z \begin{bmatrix} 0 & S_{11}^{-1} \left(T_{12} - \Phi T_{22} \right) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Z_{11}^{-1} \\ 0 \end{bmatrix} = 0 \ .$$

[10] The last step follows from $QA = S'Z$ such that $Z^{21} \varepsilon_{t+1} + Z^{22} \eta_{t+1} = 0$ where Z^{21} and Z^{22} are conformable blocks of Z', and the final result follows from $Z'Z = I$ and the partitioned inverse of Z. In this case, the invertibility of $Q_{21} A_{xy} + Q_{22} A_{yy} = S_{22} Z^{22}$ and thus the invertibility of Z_{11} is a special case of the column space condition stated in (40) of Sims (2002).

[11] Given the assumed invertibility of Z^{22}, Φ can directly be constructed from $\Phi = \left(S_{11} Z^{12} + S_{12} Z^{22} \right) \left(S_{22} Z^{22} \right)$.

References

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