International Diversification with and without Stochastic Inflation

Elmar Mertens
University of Basel, WWZ/Department of Finance
elmar.mertens@unibas.ch

October 12, 2002

Some technical things to remember: We want to . . .

- Add / subtract inflation rates and exchange rate changes from returns $\rightarrow$ Use log-changes (Why?)

- Construct portfolios, i.e. aggregate across assets
$\rightarrow$ Use simple returns (Why?)

The only way to do both, is to go to continuous time. Right now, we need this to make our approximation $1+r = \frac{1+R}{1+\pi} \rightarrow r \approx R-\pi$ work as good as it gets. Please note: We did *not* say $1+\tilde{r} = \frac{1+\tilde{R}}{1+\tilde{\pi}} \rightarrow \tilde{r} \approx \tilde{R}-\tilde{\pi}$ This will be crucial later.

Solnik-Sercu Setup

N Risky Assets N risky assets with mean returns in vector $\boldsymbol{E\tilde{R}}$ and covariance matrix $\boldsymbol{\Sigma}$. Each risky asset has portfolio weight w_i stacked in vector $\boldsymbol{w}$. They can be split into two groups: "assets" (A) and foreign money markets (C)

1 Riskfree Asset with constant return R_f. Has portfolio weight $w_f = 1 - \mathbf{1}'\boldsymbol{w}$.

No (Stochastic) Inflation It is easiest to derive the Solnik-Sercu model without any inflation. We will later see that it works also with deterministic inflation (some constant) or an inflation rate which is stochastic but uncorrelated with assets

Solnik-Sercu Solution

$$\max_{\boldsymbol{w}} V = \underbrace{\boldsymbol{w'}\boldsymbol{\mu}^e + R_f}_{\mu_p} - \frac{\eta}{2}\underbrace{\boldsymbol{w'}\boldsymbol{\Sigma}\boldsymbol{w}}_{\sigma_p}$$

$$\Rightarrow \frac{\partial V}{\partial \boldsymbol{w}} = \boldsymbol{\mu}^e - \eta\boldsymbol{\Sigma}\boldsymbol{w} \overset{!}{=} 0$$

$$\iff \boldsymbol{w} = \frac{1}{\eta}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e$$

$$\iff \boldsymbol{w} = \frac{1}{\eta}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e + \left(1 - \frac{1}{\eta}\right)\mathbf{0}$$

$$\begin{bmatrix} \boldsymbol{w} \\ w_f \end{bmatrix} = \frac{1}{\eta}\underbrace{\begin{bmatrix} \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e \\ 1 - \mathbf{1'}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e \end{bmatrix}}_{\text{“Log-Portfolio”}} + \left(1 - \frac{1}{\eta}\right)\underbrace{\begin{bmatrix} \mathbf{0} \\ 1 - \mathbf{1'}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e \end{bmatrix}}_{\text{“Hedge”}}$$

Even though the role of the “hedge” portfolio is rather void here, it mirrors the structure of the more general Adler-Dumas result

Invariance of Log-Portfolio

The log-portfolio $\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e$ is not only the portfolio held by a log-utility investor ($\eta = 1$). It is also invariant to the reference currency of the investor. The direct proof of transforming $\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}^e$ and showing that $\boldsymbol{w}$ remains the same is cumbersome (Ito Lemma and some matrix algebra, see Sercu 1980).

But we can appeal to a more illuminating argument: Maximizing log-wealth in one numeraire ($\ln W$), yields the same portfolio as maximizing log-wealth measured in another numeraire $\ln \frac{W}{S} = \ln W - \ln S$

Partitioning $\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e$

This is actually the international side of Solnik-Sercu

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{\mathsf{A,A}} & \boldsymbol{\Sigma}_{\mathsf{A,C}} \\ \boldsymbol{\Sigma}_{\mathsf{C,A}} & \boldsymbol{\Sigma}_{\mathsf{C,C}} \end{bmatrix}$$

$$\boldsymbol{B} = \boldsymbol{\Sigma}_{\mathsf{C,C}}^{-1}\boldsymbol{\Sigma}_{\mathsf{C,A}}$$

$$\boldsymbol{\Sigma}_{\mathsf{A,A|C}} = \boldsymbol{\Sigma}_{\mathsf{A,A}} - \boldsymbol{B}'\boldsymbol{\Sigma}_{\mathsf{C,C}}\boldsymbol{B}$$

$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \boldsymbol{\Sigma}_{\mathsf{A,A|C}}^{-1} & -\boldsymbol{\Sigma}_{\mathsf{A,A|C}}^{-1}\,\boldsymbol{B}' \\ -\boldsymbol{B}\,\boldsymbol{\Sigma}_{\mathsf{A,A|C}}^{-1} & \boldsymbol{\Sigma}_{\mathsf{C,C}}^{-1} + \boldsymbol{B}\,\boldsymbol{\Sigma}_{\mathsf{A,A|C}}^{-1}\boldsymbol{B}' \end{bmatrix}$$

$$\boldsymbol{w} = \begin{bmatrix} \boldsymbol{w}_{\mathsf{A}} \\ \boldsymbol{w}_{\mathsf{C}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Sigma}_{\mathsf{A,A|C}}^{-1}\,\boldsymbol{\mu}_{\mathsf{A}}^e - \boldsymbol{\Sigma}_{\mathsf{A,A|C}}^{-1}\,\boldsymbol{B}'\,\boldsymbol{\mu}_{\mathsf{C}}^e \\ \boldsymbol{\Sigma}_{\mathsf{C,C}}^{-1}\,\boldsymbol{\mu}_{\mathsf{C}}^e - \boldsymbol{B}\,\boldsymbol{w}_{\mathsf{A}} \end{bmatrix} = \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e$$

Note: $\boldsymbol{\Sigma}_{\mathsf{A,A|C}}$ is the covariance matrix of the assets *after* minimum variance hedging against currency risks

Solnik-Sercu: Numerical Example (2×2)

$$\boldsymbol{\mu}^e = \begin{bmatrix} 0.15 \\ -0.05 \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0.25^2 & 0.25 \cdot 0.20 \cdot 0.4 \\ 0.25 \cdot 0.20 \cdot 0.4 & 0.20^2 \end{bmatrix} = \begin{bmatrix} 0.25^2 & 0.02 \\ 0.02 & 0.20^2 \end{bmatrix}$$

$$\boldsymbol{B} = \frac{0.02}{0.20^2} = 0.5$$

$$\boldsymbol{\Sigma}_{\mathsf{A,A|C}} = 0.25^2 - 0.5^2 \cdot 0.20 = 0.0525$$

$$\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e = \begin{bmatrix} \frac{0.15}{0.0525} - \frac{0.5 \cdot (-0.05)}{0.0525} \\ \frac{-0.05}{0.20^2} - 0.5 \cdot \underbrace{\left(\frac{0.15}{0.0525} - \frac{0.5 \cdot (-0.05)}{0.0525} \right)}_{= w_A} \end{bmatrix} \approx \begin{bmatrix} 3.33 \\ -2.92 \end{bmatrix}$$

$$w_f = 1 - w_A - w_C \approx 1 - 3.33 + 2.92 = 0.59$$

Solnik-Sercu: Numerical Example (2×2) with Matrix Formula

Same data as before. Now we use the formula for a 2×2 inverse:

$$\boldsymbol{\mu}^e = \begin{bmatrix} 0.15 \\ -0.05 \end{bmatrix} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_A^2 & \sigma_{AB} \\ \sigma_{AB} & \sigma_B^2 \end{bmatrix} = \begin{bmatrix} 0.25^2 & 0.02 \\ 0.02 & 0.20^2 \end{bmatrix}$$

$$|\boldsymbol{\Sigma}| = \sigma_A^2 \sigma_B^2 - (\sigma_{AB})^2$$

$$\boldsymbol{\Sigma}^{-1} = \frac{1}{|\boldsymbol{\Sigma}|} \begin{bmatrix} \sigma_B^2 & -\sigma_{AB} \\ -\sigma_{AB} & \sigma_A^2 \end{bmatrix}$$

$$\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e = \underbrace{\frac{1}{0.0021} \begin{bmatrix} 0.20^2 & -0.02 \\ -0.02 & 0.25^2 \end{bmatrix}}_{=\boldsymbol{\Sigma}^{-1}} \underbrace{\begin{bmatrix} 0.15 \\ -0.05 \end{bmatrix}}_{=\boldsymbol{\mu}^e}$$

$$= \frac{1}{0.0021} \begin{bmatrix} 0.20^2 \cdot 0.15 + (-0.02) \cdot (-0.05) \\ (-0.02) \cdot 0.15 + 0.25^2 \cdot (-0.05) \end{bmatrix} \approx \begin{bmatrix} 3.33 \\ -2.92 \end{bmatrix}$$

$$w_f = 1 - w_A - w_C \approx 1 - 3.33 + 2.92 = 0.59$$

Solnik-Sercu deviation: Currency Overlay

A currency overlay proceeds sequentially, first optimizing over the *un-hedged* assets, then doing some minimum-variance currency hedging and speculating with currencies – no feedback with asset positions, though!

$$\boldsymbol{w}^{\mathsf{OV}} = \begin{bmatrix} \boldsymbol{w}_{\mathsf{A}} \\ \boldsymbol{w}_{\mathsf{C}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Sigma}_{\mathsf{A,A}}^{-1}\, \boldsymbol{\mu}_{\mathsf{A}}^{e} \\ \boldsymbol{\Sigma}_{\mathsf{C,C}}^{-1}\, \boldsymbol{\mu}_{\mathsf{C}}^{e} - \boldsymbol{B}\, \boldsymbol{w}_{\mathsf{A}} \end{bmatrix}$$

This is only optimal when $\boldsymbol{B} = \boldsymbol{0}$, then

$$\boldsymbol{w}\Big|_{\boldsymbol{B}=\boldsymbol{0}} = \boldsymbol{w}^{\mathsf{OV}}\Big|_{\boldsymbol{B}=\boldsymbol{0}} = \begin{bmatrix} \boldsymbol{w}_{\mathsf{A}} \\ \boldsymbol{w}_{\mathsf{C}} \end{bmatrix}\Bigg|_{\boldsymbol{B}=\boldsymbol{0}} = \begin{bmatrix} \boldsymbol{\Sigma}_{\mathsf{A,A}}^{-1}\, \boldsymbol{\mu}_{\mathsf{A}}^{e} \\ \boldsymbol{\Sigma}_{\mathsf{C,C}}^{-1}\, \boldsymbol{\mu}_{\mathsf{C}}^{e} \end{bmatrix}\Bigg|_{\boldsymbol{B}=\boldsymbol{0}}$$

Adler-Dumas Setup:

N Risky Assets N risky assets with *nominal* mean returns in vector $\boldsymbol{E\tilde{R}}$ and covariance matrix $\boldsymbol{\Sigma}$. Each risky asset has portfolio weight w_i stacked in vector $\boldsymbol{w}$

1 Riskfree Asset is only nominally riskfree. Return is R_f with zero variance. Note that this is a *really* risky investment! Has portfolio weight w_f. Note: $w_f = 1 - \mathbf{1'}\boldsymbol{w}$.

Inflation $\tilde{\pi}$ is stochastic with mean μ_π and variance σ_p. Covariances with the N risky assets are stacked in vector $\boldsymbol{\sigma}_{\mathsf{A}\pi}$. Note that the covariance with riskfree asset is zero (why?)

Objective:

Again we want to maximize

$$V = \mu_p - \frac{\eta}{2}\sigma_p^2$$

but now we will have stochastic inflation and need to be careful how to calculate the real returns. It is here that the trickiness of continuous time sneaks in!!

Steps:

- Figure out how to write real μ_p and σ_p^2 as function of asset weights

- Solve FOC with respect to weights to get optimal portfolio

Moments of real portfolio returns: THE WRONG WAY

$$\mu_p = \underbrace{\boldsymbol{w'E\tilde{R}} + w_f R_f - \mu_\pi}_{\text{Something missing here ...}} = \boldsymbol{w'\mu}^e + R_f - \mu_\pi$$

Where $\boldsymbol{\mu}^e \equiv \boldsymbol{E\tilde{R}} - R_f \mathbf{1}$

$$\sigma_p = \text{Var}\left(\boldsymbol{w'\tilde{R}} + w_f\, R_f - \tilde{\pi}\right) = \text{Var}\left(\boldsymbol{w'\tilde{R}} - \tilde{\pi}\right) = \boldsymbol{w'\Sigma w} - 2\boldsymbol{w'\sigma}_{\mathsf{A}\pi} + \sigma_\pi^2$$

But then we would get an inflation hedge independent from η . . .
Does the log-investor suddenly care about numeraire-risks?

$$\frac{\partial V}{\partial \boldsymbol{w}} = \boldsymbol{\mu}^e - \eta\boldsymbol{\Sigma w} + \eta\boldsymbol{\sigma}_{\mathsf{A}\pi} \stackrel{!}{=} 0 \iff \boldsymbol{w} = \frac{1}{\eta}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e + \underbrace{\boldsymbol{\Sigma}^{-1}\boldsymbol{\sigma}_{\mathsf{A}\pi}}_{\text{Inflation Hedge}}$$

DON'T DO THIS AT HOME!!!

Step 1: Real Moments

Once we have realized that we need to apply stochastic calculus in order account for *stochastic* inflation when computing the real portfolio returns, we get:

$$\mu_p = \boldsymbol{w}'\boldsymbol{\mu}^e - \underbrace{\boldsymbol{w}'\boldsymbol{\sigma}_{\mathsf{A}\pi}}_{\text{KEY!}} + R_f - \mu_\pi + \underline{\sigma_\pi^2}$$

Variance has been right:

$$\sigma_p = \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} - 2\boldsymbol{w}'\boldsymbol{\sigma}_{\mathsf{A}\pi} + \sigma_\pi^2$$

Step 2: Adler-Dumas FOC

$$\max_{\boldsymbol{w}} V = \boldsymbol{w'}\boldsymbol{\mu}^e - \boldsymbol{w'}\boldsymbol{\sigma}_{\mathsf{A}\pi} + R_f - \mu_\pi + \sigma_\pi^2 - \frac{\eta}{2}\left(\boldsymbol{w'}\boldsymbol{\Sigma}\boldsymbol{w} - 2\boldsymbol{w'}\boldsymbol{\sigma}_{\mathsf{A}\pi} + \sigma_\pi^2\right)$$

$$\Rightarrow \frac{\partial V}{\partial \boldsymbol{w}} = \boldsymbol{\mu}^e - \boldsymbol{\sigma}_{\mathsf{A}\pi} - \eta\boldsymbol{\Sigma}\boldsymbol{w} + \eta\,\boldsymbol{\sigma}_{\mathsf{A}\pi} \overset{!}{=} 0$$

$$\iff \boldsymbol{w} = \frac{1}{\eta}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}^e - \boldsymbol{\sigma}_{\mathsf{A}\pi}) + \boldsymbol{\Sigma}^{-1}\boldsymbol{\sigma}_{\mathsf{A}\pi}$$

$$\iff \boldsymbol{w} = \frac{1}{\eta}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e + \left(1 - \frac{1}{\eta}\right)\boldsymbol{\Sigma}^{-1}\boldsymbol{\sigma}_{\mathsf{A}\pi}$$

With $w_f = 1 - \boldsymbol{1'}\boldsymbol{w}$ we get Adler-Dumas formula (9)

$$\begin{bmatrix}\boldsymbol{w}\\ w_f\end{bmatrix} = \frac{1}{\eta}\underbrace{\begin{bmatrix}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e\\ 1 - \boldsymbol{1'}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e\end{bmatrix}}_{\text{``Log-Portfolio''}} + \left(1 - \frac{1}{\eta}\right)\underbrace{\begin{bmatrix}\boldsymbol{\Sigma}^{-1}\boldsymbol{\sigma}_{\mathsf{A}\pi}\\ 1 - \boldsymbol{1'}\boldsymbol{\sigma}_{\mathsf{A}\pi}\end{bmatrix}}_{\text{Inflation Hedge}}$$

With $\boldsymbol{\sigma}_{\mathsf{A}\pi} = \boldsymbol{0}$, we obtain again the Solnik-Sercu portfolio

BACKUP:
APPLICATION OF ITO LEMMA FOR ADLER DUMAS

Stochastic processes for real portfolio

N Risky Assets $dS_i = \mu_i \, S_i \, dt + \sigma_i \, S_i \, dX_i$

1 Riskfree Asset $dM = R_f \, M \, dt$

Price Level $dP = \mu_\pi \, P \, dt + \sigma_\pi \, P \, dX_P$

Where $E(dX_i) = E(dX_P) = 0$, $E(dX_i)^2 = E(dX_P)^2 = 1$, $E(dX_i dX_j) = \sigma_{ij}$ $(\forall i \neq j)$ and $E(dX_i dX_P) = \sigma_{i\pi}$

Stochastic calculus for real portfolio

We want to know how the portfolio

$$W = \frac{\sum_i n_i\, S_i + n_f\, M}{W \cdot P} \qquad \left(\text{where } w_i \equiv \frac{n_i\, S_i}{W} \cdot \frac{1}{P} \text{ and } w_f \equiv \frac{n_f\, M}{W} \cdot \frac{1}{P}\right)$$

evolves in real terms. Applying Ito's Lemma yields

$$\frac{dW}{W} = \underbrace{\left[\sum_{i=1}^{N} w_i(\mu_i - R_f - \sigma_{i\pi}) + R_f - \mu_\pi + \sigma_\pi^2\right]}_{\mu_p} dt + \underbrace{\sum_{i=1}^{N} w_i \sigma_i dX_i - \sigma_\pi dX_P}_{\sigma_p = E(\cdot)^2}$$

THIS, YOU MAY DO AT HOME!!!

. . . and you will learn how to do Ito by heart :-)

Multidimensional Ito

$$dW = \sum_{i=1}^{N} \frac{\partial W}{\partial S_i} dS_i + \frac{\partial W}{\partial M} dM + \frac{2}{2} \sum_{i=1}^{N} \frac{\partial^2 W}{\partial S_i \partial P} \sigma_{i\pi}\, S_i\, P\, dt + \frac{1}{2} \frac{\partial^2 W}{\partial P^2} \sigma_\pi^2 P^2 dt$$

$$+ \frac{\partial W}{\partial P} dP + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 W}{\partial S_i \partial S_j} \sigma_{ij}\, S_i\, S_j\, dt + \frac{1}{2} \frac{\partial^2 W}{\partial M^2} \underbrace{\sigma_M^2}_{=0} M^2 dt$$

$$+ \frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2 W}{\partial S_i \partial M} \underbrace{\sigma_{iM}}_{=0} S_i\, M\, dt + \frac{1}{2} \frac{\partial^2 W}{\partial P \partial M} \underbrace{\sigma_{\pi M}}_{=0} P\, M\, dt + \frac{\partial W}{\partial t} dt$$

$$\frac{\partial W}{\partial S_i} = \frac{n_i}{P} \qquad \frac{\partial^2 W}{\partial S_i \partial P} = -\frac{n_i}{P^2} \qquad \frac{\partial W}{\partial P} = -\frac{\sum_i n_i\, S_i + n_f\, M}{P^2}$$

$$\frac{\partial W}{\partial M} = \frac{n_f}{P} \qquad \frac{\partial^2 W}{\partial S_i \partial S_j} = 0 = \frac{\partial W}{\partial t} \qquad \frac{\partial^2 W}{\partial P^2} = 2 \cdot \frac{\sum_i n_i\, S_i + n_f\, M}{P^3}$$