The APT

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Abstract

This note summarizes matrix notation and key arguments used in the derivation of the APT as presented in class 4591, January 2002.

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1 Factor Models and Variance Decomposition

1.1 Notation

$$\tilde{\boldsymbol{R}} = \begin{bmatrix} \tilde{R}_1 \\ \tilde{R}_2 \\ \vdots \\ \tilde{R}_N \end{bmatrix}$$

$$\vec{\mu} \equiv E\left(\tilde{\boldsymbol{R}}\right) = \begin{bmatrix} E\tilde{R}_1 \\ E\tilde{R}_2 \\ \vdots \\ E\tilde{R}_N \end{bmatrix}$$

$$\tilde{\boldsymbol{F}} = \begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_2 \\ \vdots \\ \tilde{F}_F \end{bmatrix}$$

$$\tilde{\boldsymbol{\varepsilon}} = \begin{bmatrix} \tilde{\varepsilon}_1 \\ \tilde{\varepsilon}_2 \\ \vdots \\ \tilde{\varepsilon}_N \end{bmatrix}$$

Define the operator

$$\Sigma(\tilde{\boldsymbol{X}}) \equiv E\left\{\left[\tilde{\boldsymbol{X}} - E\tilde{\boldsymbol{X}}\right]\left[\tilde{\boldsymbol{X}} - E\tilde{\boldsymbol{X}}\right]'\right\} = E\left(\tilde{\boldsymbol{X}}\tilde{\boldsymbol{X}}'\right) - E\tilde{\boldsymbol{X}}E\tilde{\boldsymbol{X}}'$$

yielding the Variance-Covariance matrix of the random vector $\tilde{\boldsymbol{X}}$.

1.2 Factor Model

The multifactor model can then be written as

$$\tilde{\boldsymbol{R}} = \vec{\alpha}' + \boldsymbol{B}'\tilde{\boldsymbol{F}} + \tilde{\boldsymbol{\varepsilon}}$$

where $\vec{\alpha}$ is a row vector with typical element α_i from the i-th asset factor model $\tilde{R}_i = \alpha_i + \sum_{f=1}^{F} \beta_{f,i}\tilde{F}_f + \varepsilon_i$ and $\boldsymbol{B}$ is a $F \times N$ matrix of the factors' regression coefficients with typical element $\beta_{f,i}$. If we stack the coefficients of each factor in a row vector

$$\vec{\beta}_f = \begin{bmatrix} \beta_{f,1} & \beta_{f,f} & \cdots & \beta_{f,N} \end{bmatrix}$$

we can write

$$\boldsymbol{B} = \begin{bmatrix} \vec{\beta}_1 \\ \vec{\beta}_2 \\ \vdots \\ \vec{\beta}_F \end{bmatrix}$$

Please note that – with an eye on the APT derivation – we have consciously chosen to group the coefficients $\beta_{f,i}$ by factors not by securities. Hence our notation of a vector $\vec{\beta}_f$ does not correspond to the vector of regressions coefficients of a multivariate regression like $\tilde{y} = \boldsymbol{X}\vec{\beta} + \tilde{\varepsilon}$. Such vectors would correspond to the columns, not the rows of $\boldsymbol{B}$.

The following remarks might be skipped at first reading: Alternatively, include the 1 in the vector of factors, calling the augmented factor vector

$$\tilde{\boldsymbol{F}}_{\boldsymbol{1}} = \begin{bmatrix} 1 \\ \tilde{\boldsymbol{F}} \end{bmatrix}$$

and correspondingly

$$\boldsymbol{B}_{\boldsymbol{1}} = \begin{bmatrix} \vec{\alpha} \\ \boldsymbol{B} \end{bmatrix}$$

for the matrix of coefficients. and we can rewrite the factor model as:

$$\tilde{\boldsymbol{R}} = \boldsymbol{B}_{\boldsymbol{1}}{}'\tilde{\boldsymbol{F}}_{\boldsymbol{1}} + \tilde{\boldsymbol{\varepsilon}}$$

1.3 Variance Decomposition

The factor model regression imposes moment conditions,

$$E\left(\tilde{\boldsymbol{\varepsilon}}\right) = E\left(1\tilde{\boldsymbol{\varepsilon}}'\right) = \vec{0}$$

and

$$E\left(\tilde{\boldsymbol{F}}\tilde{\boldsymbol{\varepsilon}}'\right) = \vec{0}$$

Or written as scalars: $E\left(\tilde{\varepsilon}_i\right) = 0 \; \forall \, i$ and $E\left(\tilde{\varepsilon}_i\tilde{F}_f\right) = 0 \; \forall \, i, \; f$. From the moment conditions it follows that the variance-covariance matrix of returns can be linearly decomposed into two parts:

$$\Sigma\left(\tilde{\boldsymbol{R}}\right) = \boldsymbol{B}'\Sigma\left(\tilde{\boldsymbol{F}}\right)\boldsymbol{B} + \Sigma\left(\tilde{\boldsymbol{\varepsilon}}\right)$$

The first term in the summation represents the (co-)variation explained by the factors, the second term stands for unexplained (co-)variation of security returns. Please note that the same result is obtained using the alternative notation with $\boldsymbol{B}_{\boldsymbol{1}}$ and $\boldsymbol{F}_{\boldsymbol{1}}$ as the first row and the first column of $\Sigma\left(\boldsymbol{F}_{\boldsymbol{1}}\right)$ consist only of zeros.

1.4 Three Cases of Factor Model Restrictions

Until now, we have not imposed any assumptions on the covariance structure of returns and factors. The variance decomposition described above holds by construction. Factor models go further than that. They impose an additional assumption on the covariance structure of residual returns $\Sigma(\tilde{\boldsymbol{\varepsilon}})$. In general, there are three cases of factor models:

1.4.1 Exact Factor Model

The variance-covariance matrix of security returns is fully explained by the factor model. Each asset's regression on the factors has an R^2 of 1. This assumption is highly unrealistic, but also highly instructive for the derivation of the APT. Actually, it is the only instance where we can derive an *exact* pricing relation based on the no-arbitrage condition.

$$\Sigma(\tilde{\boldsymbol{\varepsilon}}) = \mathbf{0}$$

1.4.2 Strict Factor Model

The correlations between *different* securities are fully explained by the factor model. The residual covariance matrix is a diagonal matrix:

$$\Sigma(\tilde{\boldsymbol{\varepsilon}}) == \begin{bmatrix} \sigma^2_{\varepsilon_1} & 0 & \cdots & 0 \\ 0 & \sigma^2_{\varepsilon_2} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & \sigma^2_{\varepsilon_N} \end{bmatrix}$$

where $\boldsymbol{I}$ stands for an identity matrix with appropriate dimensions. This is what Elton and Gruber (1995, Chapters 7&8) call an "Index Model".

Market Model is no Strict Factor Model Aggregating the Market Model

$$\tilde{R}_i = \alpha_i + \beta_i \tilde{R}_M + \varepsilon_i$$

with the weights w_i^M of the market portfolio

$$\sum_{i=1}^{N} w_i^M \tilde{R}_i = \sum_{i=1}^{N} w_i^M \alpha_i + \sum_{i=1}^{N} w_i^M \beta_{M,i} \tilde{R}_M + \sum_{i=1}^{N} w_i^M \varepsilon_i$$

$$= \sum_{i=1}^{N} w_i^M E\tilde{R}_i - \sum_{i=1}^{N} w_i^M \beta_{M,i} E\tilde{R}_M + \sum_{i=1}^{N} w_i^M \beta_{M,i} \tilde{R}_M + \sum_{i=1}^{N} w_i^M \varepsilon_i$$

$$= E\tilde{R}_M - 1 \cdot E\tilde{R}_M + 1 \cdot \tilde{R}_M + \sum_{i=1}^{N} w_i^M \varepsilon_i$$

$$\Leftrightarrow \sum_{i=1}^{N} w_i^M \varepsilon_i = 0$$

shows that the Market Model cannot be an exact factor model as the last term implies that the residuals are linearly dependent (meaning that they are correlated). See also Fama (1973).

1.4.3 Approximate Factor Model

When the matrix $\Sigma(\tilde{\varepsilon})$ is "sparse", meaning that its off-diagonal are "not important" there might be chance that we get something called an approximate factor model. For instance if there is some residual correlation between stocks belonging to the same industry but no such correlation between stocks of different industries, the matrix of residual returns' covariances will have many zeros, meaning that it is sparse. Under certain conditions, an approximate factor model can do the same job for arbitrage pricing as a strict factor model. But we will not go into details here (and have remained deliberately vague) ...

1.5 Portfolio Notation

The factor model decomposes returns and their moments – in particular means and (co-)variances – into two parts. One part explained from the factors and a second part, the residuals. This carries over to portfolios of our securities. For the random return on a portfolio with weights $\vec{w}_p$ write

$$\tilde{R}_p = \vec{w}_p' \tilde{\boldsymbol{R}} = \vec{w}_p' \vec{\alpha}' + \vec{w}_p' \boldsymbol{B}' \tilde{\boldsymbol{F}} + \vec{w}_p' \tilde{\boldsymbol{\varepsilon}} = \vec{w}_p' \boldsymbol{B_1}' \tilde{\boldsymbol{F}}_{\boldsymbol{1}} + \vec{w}_p' \tilde{\boldsymbol{\varepsilon}}$$

The expected portfolio return is

$$E\tilde{R}_p = \vec{w}_p' E\left(\tilde{\boldsymbol{R}}\right) = \vec{w}_p' \vec{\alpha}' + \vec{w}_p' \boldsymbol{B}' E\left(\tilde{\boldsymbol{F}}\right) = \vec{w}_p' \boldsymbol{B_1}' E\left(\tilde{\boldsymbol{F}}_{\boldsymbol{1}}\right)$$

and the portfolio's variance (a scalar!) can be decomposed into

$$\vec{w}_p' \Sigma\left(\tilde{\boldsymbol{R}}\right) \vec{w}_p = \vec{w}_p' \boldsymbol{B}' \Sigma\left(\tilde{\boldsymbol{F}}\right) \boldsymbol{B} \vec{w}_p + \vec{w}_p' \Sigma\left(\tilde{\boldsymbol{\varepsilon}}\right) \vec{w}_p$$

2 No-Arbitrage Pricing

2.1 No-Arbitrage Conditions

No-arbitrage pricing is based on the condition that in the presence of non-satiated investors (meaning that they will *always* prefer more to less)[1], there can be no arbi-

[1] Please note that this does not imply any assumption on their risk-preferences!

trage opportunities. Formally, let us define an arbitrage portfolio[2] with weights $\vec{w}_0$ by the properties of

zero cost:

$$\vec{w}_0'\mathbf{1} = 0$$

zero downside risk, which in the presence of unbounded return distributions[3] translates to zero risk:

$$\vec{w}_0'\Sigma\left(\tilde{\boldsymbol{R}}\right)\vec{w}_0 = 0$$

Colloquially, an arbitrage opportunity can be described as "something for nothing". In our case it would mean that our arbitrage portfolio has a positive expected return $\vec{w}_0'E\left(\tilde{\boldsymbol{R}}\right)$. Please note that in the case of $\vec{w}_0'E\left(\tilde{\boldsymbol{R}}\right) < 0$, an arbitrage opportunity would arise from shorting our initial portfolio, leading to a portfolio with weights $-\vec{w}_0$. It follows the condition of no-arbitrage (NA)

zero arbitrage return:

$$\vec{w}_0'E\left(\tilde{\boldsymbol{R}}\right) = 0$$

2.2 Pricing Restriction

Using the factor model's variance decomposition, we can rewrite the zero-risk condition as

$$\vec{w}_0'\Sigma\left(\tilde{\boldsymbol{R}}\right)\vec{w}_0 = \vec{w}_0'\boldsymbol{B}'\Sigma\left(\tilde{\boldsymbol{F}}\right)\boldsymbol{B}\vec{w}_0 + \vec{w}_0'\Sigma\left(\tilde{\boldsymbol{\varepsilon}}\right)\vec{w}_0 = 0$$

2.2.1 Exact Factor Pricing

Under an exact model there is $\Sigma\left(\tilde{\boldsymbol{\varepsilon}}\right) = 0$ and the zero risk condition simplifies to

$$\vec{w}_0'\boldsymbol{B}'\Sigma\left(\tilde{\boldsymbol{F}}\right)\boldsymbol{B}\vec{w}_0 = 0$$

In order to achieve this condition, we need

$$\vec{w}_0'\boldsymbol{B}' = \begin{bmatrix}\vec{w}_0'\vec{\beta}_1 & \cdots & \vec{w}_0'\vec{\beta}_F\end{bmatrix} = \begin{bmatrix}0 & \cdots & 0\end{bmatrix} = \vec{0}$$

We see, that the three NA conditions impose orthogonality restrictions between $\vec{w}_0$ and each of the vectors $\mathbf{1}$, $\vec{\beta}_f$ ($\forall\, f = 1 \ldots F$) and $\vec{\mu}$. The APT relationship (see

[2]Please note that Ingersoll (1987) defines an arbitrage opportunity as being a zero-investment portfolio only – without regard for its riskiness.

[3]In particular if they are not bounded to be positive.

below) relies on the fact[4] that the three NA conditions imply that the vector $\vec{\mu}$ lies in the (hyper-)plane spanned by the other vectors ($\mathbf{1}$, $\vec{\beta}_f\ \forall\, f = 1 \dots F$) which are also orthogonal to $\vec{w}_0$ – see below for an illustration with $N = 3$ and $F = 1$. Hence the vector of expected returns can be written as a linear combination of the vector of ones and each of the factor coefficients vectors $\vec{\beta}_f$:

$$\vec{\mu} = \lambda_0 \mathbf{1} + \sum_{f=1}^{F} \lambda_f \vec{\beta}_f = \lambda_0 \mathbf{1} + \boldsymbol{B}\vec{\lambda} = \boldsymbol{B}_{\mathbf{1}}\vec{\lambda}_1$$

where the vectors $\vec{\lambda}$ and $\vec{\lambda}_1$ are obviously defined by

$$\vec{\lambda}_1' = \begin{bmatrix} \lambda_0 & \vec{\lambda}' \end{bmatrix} = \begin{bmatrix} \lambda_0 & \lambda_1 & \cdots & \lambda_F \end{bmatrix}$$

The orthogonality restrictions and the ensuing APT equation can easily be illustrated for $N = 3$ and $F = 1$. See figure 1. In the general case, that is for any $N < F$, we can proof[5] that the APT equation holds when we project the vector of expected returns on the vector of ones and each of the $\vec{\beta}_f$'s (Connor and Korajczyk 1995). That means we run a cross-sectional regression of expected returns on the betas including an intercept term and call the regression coefficients λ_i ($\forall\, i = 0 \dots N$):

$$\vec{\mu} = \lambda_0 \mathbf{1} + \sum_{f=1}^{F} \lambda_f \vec{\beta}_f + \vec{\eta}$$

Obviously, the only difference with the APT equation is that we allow for residuals $\vec{\eta}$. By the moment conditions of the projection (or: regression) we see that $\vec{\eta}$ has the properties of an arbitrage portfolio's weights:

$$\mathbf{1}'\vec{\eta} = 0$$

$$\vec{\beta}_f'\vec{\eta} = 0\ \forall\, f = 1 \dots F$$

Its expected return equals

$$\vec{\eta}'\mu = \vec{\eta}'\vec{\eta}$$

which must be zero by the NA condition. This is only achieved if $\vec{\eta} = \vec{0}$, which proves the APT equation.

[4] The mathematics behind this result are based on the "Fundamental Theorem of Linear Algebra". This theorem is concerned with the solutions $\vec{x}$ to $\boldsymbol{A}\vec{x} = 0$, a.k.a. the "nullspace" of $\boldsymbol{A}$. In our case $\boldsymbol{A} = \begin{bmatrix} \vec{\mu}' & \mathbf{1}' & \boldsymbol{B} \end{bmatrix}'$ is a $(F+2) \times N$ matrix and the theorem tells us that its nullspace has dimension $N - F - 2$ while the nullspace of $\begin{bmatrix} \mathbf{1}' & \boldsymbol{B} \end{bmatrix}'$ has dimension $N - F - 2$. Hence it follows that $\vec{\mu}$, who lies by construction *not* in the nullspace of $\boldsymbol{A}$ lies already in the space spanned by the columns of $\begin{bmatrix} \mathbf{1}' & \boldsymbol{B} \end{bmatrix}'$.

[5] Essentially, this is an illustration of the workings of the fundamental theorem of linear algebra.

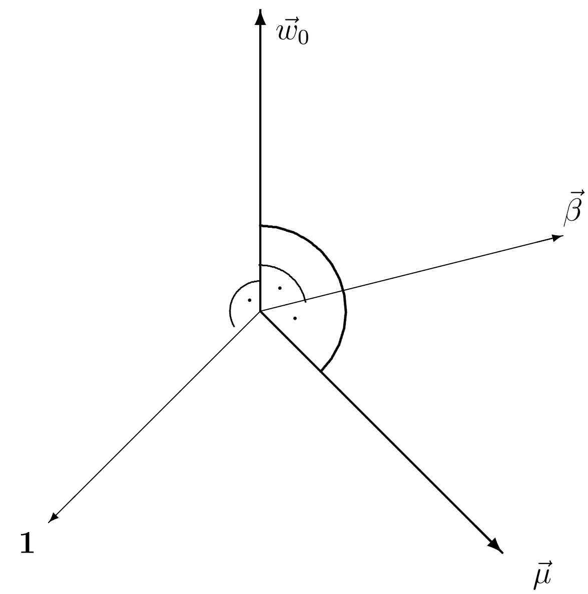

Figure 1: Orthogonality restrictions of the APT for $N = 3$ and $F = 1$

2.2.2 Approximate Factor Pricing

For large asset markets, that means $N \to \infty$, it can be shown that under a strict factor model, the residual risks can be diversified away, so that

$$\lim_{N\to\infty} \vec{w}_0' \Sigma\left(\tilde{\boldsymbol{\varepsilon}}\right) \vec{w}_0 = \lim_{N\to\infty} \sum_{i=1}^{N} w_i^2 \sigma^2(\tilde{\varepsilon}_i) \approx 0$$

now we can go back to the reasoning used in the case of an exact factor structure above and we obtain the same pricing formula, albeit only approximately.

For certain "sparse" structures of $\Sigma\left(\tilde{\boldsymbol{\varepsilon}}\right)$ – namely those leading to an approximate factor model – a similar argument can be shown to be valid. Please note that the APT holds only approximately both for strict and approximate factor structure. An exact pricing relationship is only obtained if the factor model describes the variance-covariance structure of security returns without any errors.

3 Interpreting factor premia

λ_0 is the expected return on a fully invested portfolio ($\vec{w}'\mathbf{1} = 1$) which has zero β's to each factor. In the presence of a riskfree asset, $\lambda_0 = R_F$. Each λ_f ($\forall\, f > 0$) is the expected return in excess of λ_0 on a portfolio which has $\beta_f = 1$ and $\beta_g = 0$ ($\forall\, g \neq f$).

List of Figures

References

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